THE DIFFUSION APPROXIMATION IN THREE DIMENSIONS

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7.1. INTRODUCTION AND OVERVIEW

The diffusion approximation of the radiative transport equation is used extensively because closed-form analytical solutions can be obtained. The previous chapter gave closed-form solutions to the one-dimensional diffusion equation. In this chapter, the classic searchlight problem of a finite beam of light normally incident on a slab or semi-infinite medium will be solved in the time-independent diffusion approximation. The solution follows naturally once the Green's function for the problem is known, and so the Green's function subject to homogeneous Robin boundary conditions will be given for semi-infinite and slab geometries. The diffuse radiant fluence rates are then found for impulse, flat (constant), and Gaussian shaped finite beam irradiances.

How do Green's functions help solve the problem of a finite beam incident on a turbid medium? As unscattered light propagates through the medium, it is scattered and becomes diffuse. This initial scattering event acts as a source of diffuse light. The Green's function describes the distribution resulting from a point source of diffuse light. Since the unscattered light decays exponentially with increasing depth in the slab, the Green's function for an irradiation point on the surface may be obtained by convolving the Green's function with the proper exponential function. Again using superposition, the response for an arbitrary source distribution is obtained by adding the contributions of all point irradiances. This description is not quite complete because it neglects the con-

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tribution from boundary conditions, however the analytic derivation in this chapter is complete.

The solutions for the searchlight problem are expressed as definite integrals or infinite series. There are a number of possible ways of obtaining solutions to the diffusion equation. Green's functions for a slab geometry[1] have been known for some time. Somewhat surprisingly, the Green's function for a semi-infinite medium is not readily available in the literature and is included for completeness. The solutions for the semi-infinite and slab geometries are extended to include exponentially attenuating line sources. Finally, we present equations for calculating the internal fluence rates for finite beam irradiances (flat top and Gaussian) on slab and semi-infinite media with inhomogeneous Robin boundary conditions.

To avoid the usually complicated expressions that arise in solutions for a semi-infinite geometry, some authors use monopole and dipole methods. Both techniques generate solutions that satisfy the diffusion equation at the expense of satisfying the boundary conditions. The solutions and compromises inherent in using the dipole and monopole techniques are briefly discussed.

7.2. THE PROBLEM

The problem is to find the internal distribution of light (as well as reflection and transmission) for a cylindrically symmetric beam with finite radius oriented normal to the surface of a slab or semi-infinite medium. The total power of the beam is P and the radius is w_L. The surface of the slab is flat and is unbounded in directions perpendicular to the axis of the beam. Specular reflection of the unscattered beam from the top surface is designated r_{ce} and from the bottom surface by r_{cb}. The sample is characterized by a scattering coefficient μ_s, an absorption coefficient μ_a, and an average cosine of the phase function of g. The thickness of the slab is d. The irradiance is denoted by $E(\boldsymbol{r})$ where the cylindrical coordinate system is used $\boldsymbol{r} = (r, z, \theta)$:

$$E(\boldsymbol{r}) = E(r)\exp(-\mu_t z) \tag{7.1}$$

The attenuation coefficient $\mu_t = \mu_a + \mu_s$ is the reciprocal of the average distance light travels before being scattered or absorbed by the medium. The diffuse radiant fluence rate $\phi_s(\boldsymbol{r})$ for this irradiance is the solution to the diffusion equation

$$\nabla^2\phi_s(\boldsymbol{r}) - \mu_{eff}^2\phi_s(\boldsymbol{r}) = -3\mu_s(\mu_t + g\mu_a)E(r)\exp(-\mu_t z) \tag{7.2}$$

where $\mu_{eff} = \sqrt{3\mu_a\mu_{tr}}$ is the effective attenuation coefficient and $\mu_{tr} = \mu_t -$

$g\mu_s$. The z direction coincides with the inward normal to the top surface of the slab and the boundary condition for the top boundary is

$$\phi_s(\mathbf{r}) - A\frac{\partial\phi_s(\mathbf{r})}{\partial z} = -3\mu_s gAE(\mathbf{r}) \qquad \text{at } z = 0 \tag{7.3}$$

where the boundary coefficient A is defined as

$$A = \frac{2}{3\mu_{tr}}\frac{1 + r_{21}}{1 - r_{21}} \tag{7.4}$$

The reflection factor r_{21} is discussed in the previous chapter, and represents the ratio of upward and downward hemispherical fluxes at a boundary in the diffusion approximation. The bottom boundary condition has a sign change because the inward normal is in the opposite direction,

$$\phi_s(\mathbf{r}) + A\frac{\partial\phi_s(\mathbf{r})}{\partial z} = 3\mu_s gAE(\mathbf{r}) \qquad \text{at } z = d \tag{7.5}$$

To obtain the diffuse radiant fluence rate, the inhomogeneous diffusion equation (7.2) must be solved subject to the inhomogeneous Robin boundary conditions (7.3) and (7.5). For an infinite medium, the boundary conditions are not applicable. In a semi-infinite medium (7.5) does not apply and the boundary condition instead requires that the diffuse radiant fluence rate must remain bounded at large depths.

7.2.1. Finite Beam Profiles

The functional representation for the irradiance of an impulse ring with total power P and specular reflectance r_{ce} from the surface is

$$E(r') = E_{impulse}\frac{\delta(r - r')}{r'} = (1 - r_{ce})P\frac{\delta(r - r')}{r} \tag{7.6}$$

The irradiance for a Gaussian beam with an e^{-2} radius of w_L is

$$E(r') = E_{gauss}\exp(-2r'^2/w_L^2) = \frac{4P(1 - r_{ce})}{w_L^2}\exp(-2r'^2/w_L^2) \tag{7.7}$$

The irradiance of a flat beam with radius w_L is

$$E(r') = \begin{cases} E_{flat} = \dfrac{2P(1 - r_{ce})}{w_L^2} & \text{if } r' \le w_L, \\ 0 & \text{otherwise.} \end{cases} \tag{7.8}$$

The centerline fluence rate of a Gaussian beam is twice that of a flat beam with the same total power. Furthermore, the total power

$$\int_0^{2\pi}\int_0^{\infty} E(r')\, r'dr'd\theta = (1 - r_{ce})P \tag{7.9}$$

entering the slab is equal for each of the beams [integration over azimuthal angles is not necessary since $E(r')$ is a ring source]. These irradiance expressions neglect specular reflection from the bottom boundary.

7.3. THE SOLUTION

To obtain solutions for the cylindrically symmetric three-dimensional diffusion equation in semi-infinite and slab geometries, the Green's functions are found first. This involves solving a diffusion equation with homogeneous boundary conditions and a Dirac delta function as the source term. This solution can then be used to obtain the solution for an arbitrary (cylindrically symmetric) beam profile subject to inhomogeneous boundary conditions.

7.3.1. The Green's Function

The Green's function for the diffusion equation with homogeneous boundary conditions is described by

$$\nabla^2 G(\mathbf{r};\mathbf{r}') - \mu_{eff}^2 G(\mathbf{r};\mathbf{r}') = -\delta_r(\mathbf{r} - \mathbf{r}') \tag{7.10}$$

where $\delta_r(\mathbf{r} - \mathbf{r}')$ is understood to be a ring source,

$$\delta_r(\mathbf{r} - \mathbf{r}') \equiv \frac{\delta(r - r')}{r}\,\delta(z - z') \tag{7.11}$$

The boundary condition for the top boundary is

$$G(\mathbf{r};\mathbf{r}') - A\,\frac{\partial G(\mathbf{r};\mathbf{r}')}{\partial z} = 0 \qquad \text{at } z = 0 \tag{7.12}$$

and the condition for the bottom boundary is

$$G(\mathbf{r};\mathbf{r}') + A\,\frac{\partial G(\mathbf{r};\mathbf{r}')}{\partial z} = 0 \qquad \text{at } z = d \tag{7.13}$$

The next section shows how this Green's function is related to the solution of the inhomogeneous diffusion equation (7.2) subject to inhomogeneous boundary conditions.

7.3.2. The Diffuse Radiant Fluence Rate

Green's second identity is[2]

$$\int (u\nabla^2 v - v\nabla^2 u)\, dV' = \iint \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS' \tag{7.14}$$

where n is the outward normal to the surface S' enclosing the volume V' containing all the sources. If $u = G(\boldsymbol{r};\boldsymbol{r}')$ and $v = \phi_s(\boldsymbol{r}')$, then this equation becomes

$$\int [G(\boldsymbol{r};\boldsymbol{r}')\nabla^2\phi_s(\boldsymbol{r}') - \phi_s(\boldsymbol{r}')\nabla^2 G(\boldsymbol{r};\boldsymbol{r}')]\, dV' = \int \left(G(\boldsymbol{r};\boldsymbol{r}') \frac{\partial \phi_s(\boldsymbol{r}')}{\partial n} - \phi_s(\boldsymbol{r}') \frac{\partial G(\boldsymbol{r};\boldsymbol{r}')}{\partial n} \right) dS' \tag{7.15}$$

Adding and subtracting $\int G(\boldsymbol{r};\boldsymbol{r}')\mu_{eff}^2\phi_s(\boldsymbol{r}')dV'$ from the left-hand side of this equation and then simplifying yields

$$\text{LHS} = \phi_s(\boldsymbol{r}) - 3\mu_s(\mu_t + g\mu_a) \int G(\boldsymbol{r};\boldsymbol{r}')E(r')\exp(-\mu_t z')\, dV' \tag{7.16}$$

The integral on the right-hand side can be rewritten with the stipulation that on the top surface of the slab

$$\frac{\partial}{\partial n} = -\frac{\partial}{\partial z} \quad \text{at } z = 0 \tag{7.17}$$

and

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial z} \quad \text{at } z = d \tag{7.18}$$

because z increases with depth in the slab and n is an outward normal to the slab. Upon substituting these expressions in the right-hand side we obtain

$$\begin{aligned}\text{RHS} = &- \int_{z'=0} \left(G(\boldsymbol{r};\boldsymbol{r}') \frac{\partial \phi_s(\boldsymbol{r}')}{\partial z} - \phi_s(\boldsymbol{r}') \frac{\partial G(\boldsymbol{r};\boldsymbol{r}')}{\partial z} \right) dS' \\ &+ \int_{z'=d} \left(G(\boldsymbol{r};\boldsymbol{r}') \frac{\partial \phi_s(\boldsymbol{r}')}{\partial z} - \phi_s(\boldsymbol{r}') \frac{\partial G(\boldsymbol{r};\boldsymbol{r}')}{\partial z} \right) dS'\end{aligned} \tag{7.19}$$

which simplifies, using the inhomogeneous boundary conditons (7.3), (7.5), (7.12), and (7.13), to

$$\text{RHS} = 3\mu_s g \int_{z'=0} E(r')G(\boldsymbol{r};\boldsymbol{r}')\, dS' - 3\mu_s g \exp(-\mu_t d) \int_{z'=d} E(r')G(\boldsymbol{r};\boldsymbol{r}')\, dS' \tag{7.20}$$

Equating the left-hand side, (7.16), with the right-hand side, (7.20), gives

$$\phi_s(\boldsymbol{r}) = 3\mu_s\,(\mu_t + g\mu_a) \int G(\boldsymbol{r};\,\boldsymbol{r}')E(r')\exp(-\mu_t z')\,dV' + 3\mu_s g \int_{z'=0} G(\boldsymbol{r};\,\boldsymbol{r}')E(r')\,dS' - 3\mu_s g\exp(-\mu_t d)\int_{z'=d} G(\boldsymbol{r};\,\boldsymbol{r}')E(r')\,dS' \tag{7.21}$$

The volume integral accounts for the inhomogeneous (source) term on the right-hand side of the diffusion equation, and the last two integrals arise from inhomogeneous boundary conditions at the top and bottom surfaces of the slab.

The expression for a semi-infinite medium does not have the integral for the bottom boundary

$$\phi_s(\boldsymbol{r}) = 3\mu_s(\mu_t + g\mu_a) \int G(\boldsymbol{r};\,\boldsymbol{r}')E(r')\exp(-\mu_t z')\,dV' + 2\,\mu_s g \int_{z'=0} G(\boldsymbol{r};\,\boldsymbol{r}')E(r')\,dS' \tag{7.22}$$

When scattering is isotropic, the result is

$$\phi_s(\boldsymbol{r}) = 3\mu_s\mu_t \int G(\boldsymbol{r};\,\boldsymbol{r}')E(r')\exp(-\mu_t z')\,dV' \tag{7.23}$$

because the boundary conditions become homogeneous and $\mu_{tr} = \mu_t$ when $g = 0$. Once the Green's function $G(\boldsymbol{r};\,\boldsymbol{r}')$ is known, it can be substituted into Eq. (7.21), (7.22), or (7.23) as appropriate, and the diffuse radiant fluence rate follows immediately.

7.3.3. Reflection and Transmission

Reflection and transmission are the normalized fluxes of light exiting the sample from the top and bottom. They have components consisting of unscattered (collimated) light and backscattered (diffuse) light,

$$R(r) = R_d(r) + R_c(r) \qquad \text{and} \qquad T(r) = T_d(r) + T_c(r) \tag{7.24}$$

This section gives expressions for the unscattered terms explicitly and the expressions for the diffuse terms in terms of the diffuse radiant fluence rate.

The unscattered reflection from a slab with specular reflection on the top surface of r_{ce} and specular reflection at the bottom surface of r_{cb} is

$$R_c(r) = \left[\frac{r_{ce} + r_{cb}(1 - 2r_{ce})\exp(-2\mu_t d)}{1 - r_{ce}r_{cb}\exp(-2\mu_t d)}\right]\frac{E(r)}{P} \tag{7.25}$$

The unscattered transmission is

$$T_c(r) = \frac{(1 - r_{ce})(1 - r_{cb})\exp(-\mu_t d)}{1 - r_{ce}r_{cb}\exp(-2\mu_t d)} \frac{E(r)}{P} \tag{7.26}$$

The backscattered light neglecting any internal reflection of *unscattered* light is the net diffuse flux at the surface normalized by the amount of light entering from the top,

$$R'_d(r) = -\frac{\mathbf{F}(r, z = 0) \cdot \hat{\mathbf{z}}}{(1 - r_{ce})P} \tag{7.27}$$

The negative sign ensures that the reflection is positive for light travelling in the $-z$ direction. This definition is consistent with the expression given in the previous chapter for the reflected flux (not the reflection) because

$$\begin{aligned} -\mathbf{F}(r,0) \cdot \hat{\mathbf{z}} &= F_-(0) - F_+(0) = F_-(0) - r_{21}F_-(0) \\ &= (1 - r_{21})F_-(0) \end{aligned} \tag{7.28}$$

where the definition $F(0) = F_+(0) - F_+(0)$ for the net flux and the boundary condition $F_+(0) = r_{21}F_-(0)$ at the surface were used. The transmitted diffuse light when internal reflection of unscattered light is ignored is

$$T'_d(r) = \frac{\mathbf{F}(r, z = d) \cdot \hat{\mathbf{z}}}{(1 - r_{ce})P} \tag{7.29}$$

Since a small fraction of the unscattered light can be reflected internally at the boundaries, the total diffuse reflection and transmission may differ from R'_d $(r$ and $T'_d(r)$ given above. The backscattered light from the second pass (i.e., arising only from unscattered light reflected once by only the bottom surface) is

$$R''_d(r) = r_{cb}\exp(-\mu_t d)T'_d(r) \tag{7.30}$$

Similarly, the transmitted light from the second pass is

$$T''_d(r) = r_{cb}\exp(-\mu_t d)R'_d(r) \tag{7.31}$$

The backscattered light from the third pass (i.e., arising only from unscattered light reflected once by the bottom surface and then by the top surface) is

$$R'''_d(r) = r_{ce}r_{cb}\exp(-2\mu_t d)R'_d(r) \tag{7.32}$$

Similarly, the transmitted light from the second pass is

$$T'''_d(r) = r_{ce}r_{cb}\exp(-2\mu_t d)T'_d(r) \tag{7.33}$$

Adding these and all higher-order terms yields

$$R_d(r) = \frac{R'_d(r) + T'_d(r) r_{cb} \exp(-\mu_t d)}{1 - r_{ce} r_{cb} \exp(-2\mu_t d)} \tag{7.34}$$

Similarly, the total diffuse transmission is

$$T_d = \frac{T'_d(r) + R'_d(r) r_{cb} \exp(-\mu_t d)}{1 - r_{ce} r_{cb} \exp(-2\mu_t d)} \tag{7.35}$$

To relate the diffuse reflection and transmission to the diffuse radiant fluence rate $\phi(\boldsymbol{r})$, recall that the diffuse flux (in the diffusion approximation) is

$$\boldsymbol{F}(\boldsymbol{r}) = -\frac{1}{3\mu_{tr}} \nabla\phi_s(\boldsymbol{r}) + \frac{\mu_s g}{\mu_{tr}} E(\boldsymbol{r})\, \hat{z} \tag{7.36}$$

for light incident normal to the surface. The radiant flux in the z direction is

$$\boldsymbol{F}(\boldsymbol{r}) \cdot \hat{z} = -\frac{1}{3\mu_{tr}} \frac{\partial \phi_s(\boldsymbol{r})}{\partial z} + \frac{\mu_s g}{\mu_{tr}} E(\boldsymbol{r}) \tag{7.37}$$

If the boundary condition for the surface (7.3) is used to eliminate the partial derivative, then

$$\boldsymbol{F}(r, z = 0) \cdot \hat{z} = -\frac{\phi(r,0)}{3A\mu_{tr}} \tag{7.38}$$

and therefore the first-order diffuse reflection is

$$R'_d(r) = \frac{\phi(r,0)}{3A\mu_{tr}(1 - r_{ce})P} = \frac{1}{2}\left(\frac{1 - r_{21}}{1 + r_{21}}\right) \frac{\phi(r,0)}{(1 - r_{ce})P} \tag{7.39}$$

If the bottom boundary condition is used, then the first-order diffuse transmission is

$$T'_d(r) = \frac{\phi(r,d)}{3A\mu_{tr}(1 - r_{ce})P} = \frac{1}{2}\left(\frac{1 - r_{21}}{1 + r_{21}}\right) \frac{\phi(r,d)}{(1 - r_{ce})P} \tag{7.40}$$

The total diffuse reflection for a slab is

$$R_d(r) - \frac{1}{2}\left(\frac{1 - r_{21}}{1 + r_{21}}\right) \frac{\phi(r,0) + \phi(r,d) r_{cb} \exp(-\mu_t d)}{(1 - r_{ce})P[1 - r_{ce} r_{cb} \exp(-2\mu_t d)]} \tag{7.41}$$

The total diffuse transmission is

$$T_d(r) = \frac{1}{2}\left(\frac{1 - r_{21}}{1 + r_{21}}\right) \frac{\phi(r,d) + \phi(r,0) r_{cb} \exp(\mu_t d)}{(1 - r_{ce})P[1 - r_{ce} r_{cb} \exp(-2\mu_t d)} \tag{7.42}$$

Typically, the corrections resulting from including multiple internal reflections are small since r_{cb} and r_{ce} are on the order of 4%. It is noteworthy that if $d \to \infty$ (semi-infinite medium) or if $r_{cb} = 0$, then

$$R_d(r) = \frac{1}{2}\left(\frac{1 - r_{21}}{1 + r_{21}}\right)\frac{\phi(r, 0)}{(1 - r_{ce})P} \tag{7.43}$$

Once an expression for the diffuse radiant flux has been obtained, then the reflection and transmission follow immediately from the above relations.

7.4. DIFFUSE RADIANT FLUENCE RATE IN A SEMI-INFINITE MEDIUM

Two different methods of obtaining the diffuse radiant fluence rate in a semi-infinite medium are presented here. The first finds the Green's function and convolves this with the incident beam distribution to get the diffuse radiant fluence rate. The second method uses monopole and dipole sources to obtain solutions to the diffusion equation that approximately satisfy the boundary conditions.

7.4.1. The Green's Function for a Ring Source

The Green's functions for a ring source in a semi-infinite medium are found by first solving the radial part of the problem and then proceeding to extract the axial component of the solution. This Green's function is then convolved to obtain solutions for a ring impulse, a flat beam, and a Gaussian beam.

7.4.1.1. Radial Solution

Assume that the Green's function will have the form (in what is essentially a Hankel transform)

$$G(r, z; r', z') = \int_0^\infty sJ_0(rs)g(s,z; r', z')\, ds \tag{7.44}$$

Substituting this expression into the diffusion equation yields

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\int_0^\infty sJ_0(rs)g(s,z; r', z')\, ds\right) + \int_0^\infty sJ_0(rs)\left[\frac{\partial^2}{\partial z^2} - \mu_{eff}^2\right] g(s,z; r', z')\, ds = -\frac{\delta(r - r')}{r}\delta(z - z') \tag{7.45}$$

The definition of the zero-order Bessel function can be used to obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} J_0(rs)\right) = -s^2 J_0(rs) \tag{7.46}$$

Using this result, the diffusion equation simplifies to

$$\int_0^\infty sJ_0(r,s)\left[\frac{\partial^2}{\partial z^2} - (s^2 + \mu_{eff}^2)\right] g(s,z;\, r',\, z')\, ds = -\frac{\delta(r - r')}{r}\delta(z - z') \tag{7.47}$$

The orthogonality relation of Bessel functions

$$\int_0^\infty sJ_0(rs)\, J_0(r's)\, ds = \frac{\delta(r - r')}{r} \tag{7.48}$$

When this is multiplied by $-\delta(z - z')$,

$$\int_0^\infty sJ_0(rs)\, [-\delta(z - z')J_0(r'\, s)]\, ds = -\frac{\delta(r - r')}{r}\delta(z - z') \tag{7.49}$$

By equating Eqs. (7.47) and (7.49), we obtain

$$\int_0^\infty sJ_0(rs)\left[\left(\frac{\partial^2}{\partial z^2} - (s^2 + \mu_{eff}^2)\right) g(s,z;\, r',\, z')\right] ds = \int_0^\infty sJ_0(rs)\, [\, -\delta(z - z')\, J_0(r's)]\, ds \tag{7.50}$$

Now since these must be satisfied for all positive s, the bracketed quantities must be equal and the following differential equation in z is obtained:

$$\frac{\partial^2}{\partial z^2} g(s,z;\, r',\, z') - (s^2 + \mu_{eff}^2) g(s,z;\, r',\, z') = -\delta(z - z')J_0(r's) \tag{7.51}$$

7.4.1.2. Axial Solution

To simplify notation, let

$$f(z;\, z') = g(s,z;\, r',\, z') = -J_0(r's)f(z;\, z') \qquad \text{and} \qquad \alpha^2 = s^2 + \mu_{eff}^2 \tag{7.52}$$

Substituting this into Eq. (7.51) yields the following one-dimensional differential equation:

$$f''(z) - \alpha^2 f(z;\, z') = \delta(z - z') \tag{7.53}$$

that must be solved subject to the homogeneous Robin boundary condition,

$$f(z; z') - Af'(z) = 0 \qquad \text{at } z = 0 \tag{7.54}$$

We assume a solution of the form

$$f(z;z') = \begin{cases} A_1 e^{\alpha(z - z')} + B_1 e^{-\alpha(z - z')}, & \text{if } z > z' \\ A_2 e^{\alpha(z - z')} + B_2 e^{-\alpha(z - z')}, & \text{if } z < z' \end{cases} \tag{7.55}$$

We observe that both solutions satisfy the homogeneous differential equation

$$f''(z; z') - \alpha^2 f(z; z') = 0 \tag{7.56}$$

when $z \neq z'$. To find the undetermined coefficients we use the restrictions of the Green's functions and the boundary conditions.

The requirement for continuity at $z = z'$ requires that

$$A_1 + B_1 = A_2 + B_2 \tag{7.57}$$

The jump requirement in the first derivative of the one-dimensional Green's function $f(z;z')$

$$\left. \frac{df(z; z')}{dz} \right|_{z \to z'^-}^{z \to z'^+} = 1 \tag{7.58}$$

results from the delta-function source at z' and requires

$$A_1 - B_1 - A_2 + B_2 = \frac{1}{\alpha} \tag{7.59}$$

The requirement that the solution is bounded at $z \to \infty$ requires

$$A_1 = 0 \tag{7.60}$$

The boundary condition at $z = 0$ requires

$$A_2 e^{-\alpha z'} + B_2 e^{\alpha z'} - \alpha A A_2 e^{-\alpha z'} + \alpha A B_2 e^{\alpha z'} = 0 \tag{7.61}$$

Three equations in three unknowns are solved to get the final solution,

$$f(z; z') = \frac{1}{2\alpha(1 + \alpha A)} \begin{cases} (1 - \alpha A)e^{-\alpha(z+z')} - (1 + \alpha A)e^{-\alpha(z-z')} & \text{if } z > z' \\ (1 - \alpha A)e^{-\alpha(z+z')} - (1 + \alpha A)e^{-\alpha(z-z')} & \text{if } z < z' \end{cases} \tag{7.62}$$

7.4.1.3. The Solution

The Green's function for a ring of sources with radius $\boldsymbol{r}'$ at a depth z' in a semi-infinite medium with homogeneous boundary conditions is

$$G(r,z;\ r',z') = \int_0^\infty J_0(rs)g(s,z;\ r',z')\ s\ ds \tag{7.63}$$

where $g(s,z;\ r',\ z')$ can be expressed explicitly using Eqs. (7.52) and (7.62) as

$$g(s,\ z;\ r',\ z') = \frac{-J_0(r'\ s)}{2\alpha(1+\alpha A)}\begin{cases}(1-\alpha A)e^{-\alpha(z+z')}-(1+\alpha A)e^{-\alpha(z-z')} & \text{if } z > z' \\ (1-\alpha A)e^{-\alpha(z+z')}-(1+\alpha A)e^{-\alpha(z-z')} & \text{if } z < z'\end{cases} \tag{7.64}$$

where $\alpha^2 = s^2 + \mu_{eff}^2$.

Recall that the normalization for this Green's function is

$$\int_0^\infty \int_0^\infty G(r,z;\ r',z')\ r'dr'dz' \tag{7.65}$$

and integration about the azimuthal angle is not necessary.

7.4.1.4. The Green's Function for an Extended Ring Source

The Green's function for an exponentially decaying line of sources located at r' and extending perpendicular to the surface of the semi-infinite medium can be expressed as

$$G_\ell(r,z;r') = \int_0^\infty G(r,z;\ r',z')\exp(-\mu_t z')\ dz' \tag{7.66}$$

where the subscript ℓ indicates that this is the Green's function for a "line" of sources (actually, a cylindrical shell of sources). This may be rewritten as

$$G_\ell(r,z;r') = \int_0^\infty J_0(rs)g(s,z;\ r')\ s\ ds \tag{7.67}$$

where

$$g(s,z;r') = \int_0^\infty g(r,z;\ r',\ z')\exp(-\mu_t z')\ dz' \tag{7.68}$$

Substituting Eq. (7.64) into (7.68) and integrating over z' gives

$$g(s,z;\ r') = \frac{J_0(r's)}{s^2+\mu_{eff}^2-\mu_t^2}\left[\exp(-\mu_t z) - \frac{1+\mu_t A}{1+\alpha A}\exp(-\alpha z)\right] \tag{7.69}$$

where

$$s^2 = \alpha^2 - \mu_{eff}^2 \tag{7.70}$$

The Green's function for a ring of exponentially decaying sources is

$$G_\ell(r,z;\, r') = \int_0^\infty \frac{J_0(r,s)J_0(r's)}{s^2 + \mu_{eff}^2 - \mu_t^2}\left[\exp(-\mu_t z) - \frac{1 + \mu_t A}{1 + \alpha A}\exp(-\alpha z)\right] sds \tag{7.71}$$

7.4.2. Diffuse Radiant Fluence Rates for Finite Beams

To find the diffuse radiant fluence rate in a semi-infinite medium we must evaluate Eq. (7.22),

$$\phi_s(r,z) = 3\mu_s(\mu_t + g\mu_a)\int G(\mathbf{r};\mathbf{r}')E(r')\exp(-\mu_t z')\,dV' + 3\mu_s g\int_{z'=0} G(\mathbf{r};\mathbf{r}')E(r')\,dS' \tag{7.72}$$

For convenience this will be rewritten as

$$\phi_s(r,z) = 3\mu_s(1 - r_{ce})P[(\mu_t + g\mu_a)I_V(r,z) + gI_S(r,z)] \tag{7.73}$$

where I_V is the volume integral and I_S is the surface (boundary condition) integral. The volume integral is dimensionless, but the surface integral has dimensions of inverse length.

The volume integral is

$$\begin{aligned}
I_V(r,z) &= \int G(\mathbf{r};\mathbf{r}')\frac{E(\mathbf{r}')}{(1 - r_{ce})P}\,dV' \\
&= \int_0^\infty\int_0^\infty G(r,z;r',z')\frac{E(r')}{(1 - r_{ce})P}\exp(-\mu_t z')\,r'dr'dz' \\
&= \int_0^\infty G_\ell(r,z;r')\frac{E(r')}{(1 - r_{ce})P}\,r'dr' \\
&= \int_0^\infty \frac{J_0(rs)}{s^2 + \mu_{eff}^2 - \mu_t^2}\left\{\int_0^\infty J_0(r's)\frac{E(r')}{(1 - r_{ce})P}\,r'dr'\right\} \\
&\qquad \left[\exp(-\mu_t z) - \frac{1 + \mu_t A}{1 + \alpha A}\exp(-\alpha z)\right] sds \\
&= \int_0^\infty \frac{J_0(rs)H(s)}{s^2 + \mu_{eff}^2 - \mu_t^2}\left[\exp(-\mu_t z) - \frac{1 + \mu_t A}{1 + \alpha A}\exp(-\alpha z)\right] sds
\end{aligned} \tag{7.74}$$

where the integral in braces has been denoted by

$$H(s) = \int_0^\infty J_0(r's) \frac{E(r')}{(1 - r_{ce})P} r' dr' \tag{7.75}$$

The surface integral is

$$\begin{aligned} I_S(r,z) &= \int G(\mathbf{r}; \mathbf{r}') \frac{E(\mathbf{r}')}{(1 - r_{ce})P} dS' \qquad \text{at } z' = 0 \\ &= \int_0^\infty G(r,z; r',z' = 0) \frac{E(r')}{(1 - r_{ce})P} r' \, dr' \end{aligned} \tag{7.76}$$

Now $z' = 0$ implies $z > z'$, and therefore $g(s,z; r',z')$ in Eq. (7.64) is

$$g(s,z; r',0) = A \frac{J_0(r's)}{1 + \alpha A} \exp(-\alpha z) \tag{7.77}$$

The surface integral is

$$\begin{aligned} I_S(r,z) &= A \int_0^\infty \int_0^\infty \frac{J_0(rs) J_0(r's)}{1 + \alpha A} \frac{E(r')}{(1 - r_{ce})P} \exp(-\alpha z) \, r' dr' \, s ds \\ &= A \int_0^\infty \frac{J_0(rs) H(s) \exp(-\alpha z)}{1 + \alpha A} s ds \end{aligned} \tag{7.78}$$

In the next three sections, impulse, flat, Gaussian profiles for $E(r')$ will be substituted into Eq. (7.75) to obtain explicit expressions for I_V and I_S.

7.4.2.1. Impulse Beam

The integral $H(s)$ for an impulse ring located at a radius w_L is

$$\begin{aligned} H(s) &= \frac{E_{impulse}}{(1 - r_{ce})P} \int_0^\infty J_0(r's) \frac{\delta(w_L - r')}{r'} r' dr' \\ &= J_0(w_L s) \end{aligned} \tag{7.79}$$

The volume integral $I_V(r,z)$ is then

$$I_V(r,z) = \int_0^\infty \frac{J_0(rs) J_0(w_L s)}{s^2 + \mu_{eff}^2 - \mu_t^2} \left[\exp(-\mu_t z) - \frac{1 + \mu_t A}{1 + \alpha A} \exp(-\alpha z) \right] s ds \tag{7.80}$$

The surface integral $I_S(r,z)$ is given by

$$I_S(r,z) = A \int_0^\infty \frac{J_0(rs) J_0(w_L s)}{1 + \alpha A} \exp(-\alpha z) \, s ds \tag{7.81}$$

7.4.2.2. Flat Beam

The inner integral for a flat beam with radius w_L is

$$
\begin{aligned}
H(s) &= \frac{E_{flat}}{(1 - r_{ce})P} \int_0^{w_L} J_0(r's)\, r'dr' \\
&= \frac{2J_1(w_L s)}{w_L s}
\end{aligned}
\tag{7.82}
$$

using equation 6.561.5 from Gradshteyn and Ryzhik.[3] The volume integral for a flat beam is

$$
I_V(r,z) = 2\int_0^{\infty} \frac{J_0(rs)J_1(w_L s)}{w_L(s^2 + \mu_{eff}^2 - \mu_t^2)} \left[\exp(-\mu_t z) - \frac{1 + \mu_t A}{1 + \alpha A} \exp(-\alpha z) \right] ds
\tag{7.83}
$$

The surface integral is

$$
I_S(r,z) = 2A \int_0^{\infty} \frac{J_0(rs)J_1(w_L s)}{w_L(1 + A\alpha)} \exp(-\alpha z)\, ds
\tag{7.84}
$$

7.4.2.3. Gaussian Beam

The inner integral for a Gaussian beam with radius w_L is

$$
\begin{aligned}
H(s) &= \frac{E_{gauss}}{(1 - r_{ce})P} \int_0^{\infty} J_0(r's) \exp[-2(r'/w_L)^2] r'dr' \\
&= \exp(-s^2 w_L^2/8)
\end{aligned}
\tag{7.85}
$$

using equation 6.631.5 from Gradshteyn and Ryzhik.[3] The volume integral for a Gaussian beam is

$$
I_V(r,z) = \int_0^{\infty} \frac{J_0(rs) \exp(-s^2 w_L^2/8)}{s^2 + \mu_{eff}^2 - \mu_t^2} \left[\exp(-\mu_t z) - \frac{1 + \mu_t A}{1 + \alpha A} \exp(-\alpha z) \right] sds
\tag{7.86}
$$

We note in passing that this result is identical to that obtained by Grossweiner *et al.*,[4] who do not include the surface term since homogeneous boundary conditions were used. The surface integral is

$$
I_S(r,z) = A \int_0^{\infty} \frac{J_0(rs) \exp(-s^2 w_L^2/8)}{1 + \alpha A} \exp(-\alpha z)\, sds
\tag{7.87}
$$

The difficulty with these approaches is that the proper boundary conditions for light transport in a turbid medium in the diffusion approximation require that homogeneous Robin boundary conditions, Eq. (3), must be satisfied. Neither of the simple techniques described above can be extended to the Robin boundary condition case. This is readily shown using the explicit forms for the unbounded Green's functions.

The boundary condition at the surface

$$G(\boldsymbol{r};\boldsymbol{r}') = A\,\frac{\partial G(\boldsymbol{r};\boldsymbol{r}')}{\partial z} \qquad \text{at } z = 0 \tag{7.99}$$

states that the first derivative of the desired Green's function is proportional to itself. The first derivative of $G_m(\boldsymbol{r};\boldsymbol{r}')$ can be written

$$\frac{\partial G_m(\boldsymbol{r};\boldsymbol{r}')}{\partial z} = -\frac{z-z'}{(\Delta r)^2}(1+\Delta r)G_m(\boldsymbol{r};\boldsymbol{r}') \tag{7.100}$$

If the first derivative is evaluated at $z = 0$,

$$\frac{\partial G_m(r,0,\theta;\,r',z',\theta')}{\partial z} = z'\left[\frac{1+\Delta r}{(\Delta r)^2}\right]G_m(r,0,\theta;\,r',z',\theta') \tag{7.101}$$

we find that the first derivative and the function itself are not related by a constant factor. Therefore, no finite linear combination of virtual image sources will suffice to satisfy the homogeneous Robin boundary conditions that arise in this problem.

There have been a few attempts at resolving this problem. The first is to ignore the Robin conditions and use Dirichlet boundary conditions. In this case, monopoles are located at $\pm z'$ as proposed by Patterson *et al.*,[8]

$$G(\boldsymbol{r};\boldsymbol{r}') = G_m(r,z,\theta;\,0,1/\mu_s',0) - G_m(r,z,\theta;\,0,-1/\mu_s',0) \tag{7.102}$$

where the sources are located on the centerline of the beam at a depth of $z' = 1/\mu_s'$. Later, Farrell *et al.* moved the sources slightly to obtain

$$G(\boldsymbol{r};\boldsymbol{r}') = G_m(r,z,\theta;\,0,1/\mu_t',0) - G_m(r,z,\theta;\,0,-1/\mu_t',0) \tag{7.103}$$

An equivalent technique involving dipoles is to simply place the dipole on the $z = 0$ plane,

$$G(r,z,\theta;\,r',z',\theta') = G_d(r,z,\theta;\,0,0,0) \tag{7.104}$$

This was the early approach by Fretterd and Longini.[5]

A better approximation of the Robin boundary conditions results when the external source is placed at a point other than the mirror image of the internal source. This approach was used by Farrell *et al.*,[10] who cite Moulton[13] as their reason for choosing two monopoles located with an offset of A. Farrell's sources were located at $z' = -\mu_t'^{-1} - 2A$ and the internal source was still at $z = \mu_t'^{-1}$ to obtain

$$G(r; r') = G_m(r,z,\theta; 0, 1/\mu_t', 0) - G_m(r,z,\theta; 0, -1/\mu_t' - 2A, 0) \tag{7.105}$$

A similar argument was used by Allen *et al.*[9] in the dipole approximation who noted the functional dependence on the radial components. They went on to observe that for large radii, the functional dependence becomes a constant and that the dipole should be located at $z' = -A$ to satisfy the asymptotic boundary conditions. Allen also goes on to find a relation for the magnitude of the dipole moment and proposed the modified diffusion dipole equation

$$\phi_s(r,z) = \frac{2\mu_s P(2z + A)\exp(\mu_{eff}A)}{(1 + \mu_{eff}A)(\Delta r)^3}(1 + \mu_{eff}\Delta r)\exp(-\mu_{eff}\Delta r) \tag{7.106}$$

It should be noted that all attempts so far to resolve the boundary condition problem using the method of images have been only approximate solutions. Experimental and Monte Carlo simulations have verified that some of these approaches give quite acceptable answers at sufficient distances from the beam. It is also noteworthy that if inhomogeneous boundary conditions (i.e., those that hold for anisotropic media) are to be simulated, then each of the expressions given in this section must be substituted into the Green's function formalism to include the effects of inhomogeneous boundaries. If specific beam shapes are desired, then convolution must be performed over the beam shape. This was done by Eason *et al.*[7] Unfortunately, this process destroys the primary attraction of the monopole and dipole approximations—analytic simplicity. In fact, the integrals that result are worse than those obtained using the Green's function approach above.

7.5. DIFFUSE RADIANT FLUENCE RATE IN A SLAB

7.5.1. The Green's Function for a Ring Source

The Green's function for a cylindrically symmetric ring source in a slab with thickness d at a radius r' and depth z' that satisfies the differential equation (7.10) is[1]

$$G(r,z; r', z') = \sum_{n=1}^{\infty} \frac{\sin(k_n z + \gamma_n)\sin(k_n z' + \gamma_n)}{N_n} \begin{cases} K_0(\lambda_n r) I_0(\lambda_n r'), & \text{if } r > r' \\ K_0(\lambda_n r') I_0(\lambda_n r), & \text{otherwise} \end{cases} \tag{7.107}$$

where K_0 and I_0 are modified Bessel functions. The restriction on the eigenvalues γ_n is obtained by substituting the Green's function into the boundary condition (7.12) at $z = 0$,

$$\tan \gamma_n = Ak_n \tag{7.108}$$

The equation governing the eigenvalues k_n is obtained by imposing the boundary condition (7.13) at $z = d$ and simplifying,

$$\tan(k_n d + \gamma_n) = -Ak_n \tag{7.109}$$

Using (7.108) and some trigonometry, we obtain

$$\tan(k_n d) = \frac{2Ak_n}{A^2k_n^2 - 1} \tag{7.110}$$

The normalization factor N_n is given by

$$N_n = \int_0^d \sin^2(k_n z + \gamma_n)\, dz = \frac{\sin 2\gamma_n - \sin(2k_n d + 2\gamma_n) + 2k_n d}{4k_n} \tag{7.111}$$

Finally, substituting the Green's function (7.107) into the diffusion equation (7.2) results in a relation between λ_n and k_n:

$$\lambda_n^2 = k_n^2 + \mu_{eff}^2 \tag{7.112}$$

Now if the Green's function is convolved with a decaying exponential over the depth of the slab, then response for an extended ring source is obtained:

$$G_\ell(r,z;\, r') = \int_0^d G(r,z;\, r',z') \exp(-\mu_t z')\, dz' \tag{7.113}$$

Substituting and integrating reveals that the extended source Green's function is

$$G_\ell(r,z;\, r') = \sum_{n=1}^{\infty} \frac{\sin(k_n z + \gamma_n)}{N_n} \frac{z_n}{k_n^2 + \mu_t^2} \begin{cases} K_0(\lambda_n r) I_0(\lambda_n r'), & \text{if } r > r' \\ K_0(\lambda_n r') I_0(\lambda_n r), & \text{otherwise} \end{cases} \tag{7.114}$$

where z_n is given by

$$\begin{aligned} z_n = {} & \sin \gamma_n \, [\mu_t + \exp(-\mu_t d)\, (k_n \sin k_n d - \mu_t \cos k_n d)] \\ & + \cos \gamma_n \, [k_n - \exp(-\mu_t d)\, (\mu_t \sin k_n d + k_n \cos k_n d)] \end{aligned} \tag{7.115}$$

7.5.2. Diffuse Radiant Fluence Rates for Finite Beams

For a slab geometry, the volume integral only extends from 0 to d and the surface integral must also include the bottom boundary. The expression for the diffuse radiant fluence rate (7.21) can be separated into three terms,

$$\phi_d(r,z) = 3\mu_s(1 - r_{ce})P[(\mu_t + g\mu_a)I_V(r,z) + gI_{top}(r,z) - g\exp(-\mu_t d)I_{bottom}] \tag{7.116}$$

The volume integral is given by

$$\begin{aligned} I_V(r,z) &= \int G(\mathbf{r};\mathbf{r}')\frac{E(\mathbf{r}')}{(1 - r_{ce})P} \\ &= \int_0^\infty G_\ell(r,z;r')\frac{E(r')}{(1 - r_{ce})P}r'dr' \\ &= \sum_{n=1}^{\infty}\frac{\sin(k_n z + \gamma_n)}{N_n}\frac{z_n}{k_n^2 + \mu_t^2}H_n(r) \end{aligned} \tag{7.117}$$

where the radial integral has been denoted by

$$H_n(r) = K_0(\lambda_n r)\int_0^r I_0(\lambda_n r')\frac{E(r')}{(1 - r_{ce})P}r'dr' + I_0(\lambda_n r)\int_0^\infty K_0(\lambda_n r')\frac{E(r')}{(1 - r_{ce})P}r'dr' \tag{7.118}$$

The surface integral for the top is given by

$$\begin{aligned} I_{top}(r,z) &= \int_0^\infty G(r,z;r',0)\frac{E(r')}{(1 - r_{ce})P}r'dr' \\ &= \sum_{n=1}^{\infty}\frac{\sin\gamma_n\sin(k_n z + \gamma_n)}{N_n}H_n(r) \end{aligned} \tag{7.119}$$

The surface integral for the bottom is

$$\begin{aligned} I_{bottom}(r,z) &= \int_0^\infty G(r,z;r',d)\frac{E(r')}{(1 - r_{ce})P}r'dr' \\ &= \sum_{n=1}^{\infty}\frac{\sin(k_n z + \gamma_n)\sin(k_n d + \gamma_n)}{N_n}H_n(r) \end{aligned} \tag{7.120}$$

Now, based on the restrictions on the eigenvalues (7.108) and (7.109), the following relations hold:

$$\sin\gamma_n = \frac{Ak_n}{\sqrt{A^2k_n^2 + 1}} = -\sin(k_n d + \gamma_n) \tag{7.121}$$

The top and bottom surface integral relations can be combined to obtain

$$I_S(r,z) = I_{top}(r,z) - \exp(-\mu_t d) I_{bottom}(r,z)$$
$$= \sum_{n=1}^{\infty} \frac{\sin\gamma_n \sin(k_n z + \gamma_n) z_n}{N_n} [1 + \exp(-\mu_t d)]\, H_n(r) \tag{7.122}$$

The diffuse radiant fluence rate now has the same form as in the semi-infinite case:

$$\phi_s(r,z) = 3\mu_s(1 - r_{ce})P[(\mu_t + g\mu_a)I_V(r,z) + gI_S(r,z)] \tag{7.123}$$

It just remains to evaluate $H_n(r)$ for different beam distributions.

7.5.2.1. Impulse Ring

The radial integral for a ring source with radius w_L is

$$H_n(r) = K_0(\lambda_n r)\int_0^r I_0(\lambda_n r')\delta(r' - w_L)dr' + I_0(\lambda_n r)\int_r^{\infty} K_0(\lambda_n r')\delta(r' - w_L)\, dr'$$
$$= \begin{cases} K_0(\lambda_n r)I_0(\lambda_n w_L), & \text{if } r > w_L \\ K_0(\lambda_n w_L)I_0(\lambda_n r), & \text{otherwise} \end{cases} \tag{7.124}$$

Therefore, the volume integral is

$$I_V(r,z) = \sum_{n=1}^{\infty} \frac{\sin(k_n z + \gamma_n) z_n}{(k_n^2 + \mu_t^2)N_n} \begin{cases} K_0(\lambda_n r)I_0(\lambda_n w_L), & \text{if } r > w_L \\ K_0(\lambda_n w_L)I_0(\lambda_n r), & \text{otherwise} \end{cases} \tag{7.125}$$

The surface integral is

$$I_S(r,z) = [1 + \exp(-\mu_t d)] \sum_{n=1}^{\infty} \frac{\sin\gamma_n \sin(k_n z + \gamma_n)}{N_n} \begin{cases} K_0(\lambda_n r)I_0(\lambda_n w_L), & \text{If } r > w_L \\ K_0(\lambda_n w_L)I_0(\lambda_n r), & \text{otherwise} \end{cases} \tag{7.126}$$

7.5.2.2. Flat Beam

The radial integral for a flat beam with radius w_L is

$$H_n(r) = \frac{2}{w_l^2}\left[K_0(\lambda_n r)\int_0^r I_0(\lambda_n r')\, r'dr' + I_0(\lambda_n r)\int_r^{w_L} K_0(\lambda_n r')\, r'dr' \right]$$
$$= \frac{2}{w_L^2\lambda_n^2} \begin{cases} \lambda_n w_L K_0(\lambda_n r)I_1(\lambda_n w_L), & \text{if } r > w_L \\ 1 - \lambda_n w_L K_1(\lambda_n w_L)I_0(\lambda_n r), & \text{otherwise} \end{cases} \tag{7.127}$$

This leads to a solution for the diffuse radiant fluence that is equivalent to Reynolds's solution[1] except that Reynolds neglected the inhomogeneous boundary condition at the surface.

7.5.2.3. Gaussian

The inner integral for a Gaussian beam with radius w_L is

$$\begin{aligned} H_n(r) &= \frac{4}{w_L^2} K_0(\lambda_n r) \int_0^r I_0(\lambda_n r') \exp[-2(r'/w_L)^2] r' dr' \\ &+ \frac{4}{w_L^2} I_0(\lambda_n r) \int_r^\infty K_0(\lambda_n r') \exp[-2(r'/w_L)^2] r' dr' \end{aligned} \tag{7.128}$$

which has not been solved analytically.

7.6. DISCUSSION

The intent of this chapter has been to give complete accurate analytical expressions for the diffuse radiant fluence rate for the searchlight problem on flat-slab and semi-infinite geometries. Special care was taken to incorporate the entire boundary condition for the interface between a scattering medium and a nonscattering medium with different indices of refraction (i.e., the inhomogeneous Robin condition) so that when the delta-Eddington approximation[14] is used the boundary conditions remain accurate. Usually, homogeneous boundary conditions are used because the similarity principle is used to convert the anisotropic problem to an isotropic problem [$\mu_s \rightarrow \mu_s(1 - g)$ and $g \rightarrow 0$]. Once the problem is isotropic, the boundary conditions become homogeneous. Unfortunately, this is not as accurate as the delta-Eddington approximation.[15]

Included also are the formulas for the reflection and transmission in which multiple internal reflections of unscattered light are incorporated accurately. Including these interactions may seem incongruous with the implicit inaccuracies of the diffusion approximation, but it yields answers that are exactly correct when $\mu_s = 0$ and gives much better approximations when the unscattered light is much larger than the scattered component. Finally, the expressions for the diffuse reflection and diffuse transmission are given in terms of the diffuse radiant fluence rate. This allows the rest of the chapter to focus on finding expressions for the diffuse radiant fluence rate with no need to find separate expressions for the radial dependence of the diffuse reflection and transmission.

Expressions for the diffuse radiant fluence rate in a semi-infinite geometry are given for a ring of irradiance, uniform irradiance over a finite beam, and

Gaussian irradiance. These solutions incorporate inhomogeneous Robin boundary conditions, and should be applicable to media with anisotropic scattering and mismatched indices of refraction at the boundaries. Unfortunately, the expressions are in terms of integrals of Bessel functions. Since these expressions involve only a single integration, they are amenable to numerical integration.

The monopole and dipole approximations for the diffuse radiant fluence rate offer analytic simplicity and have been shown in the literature to work reasonably well at larger radii. However, the fact that the boundary conditions are homogeneous makes these solutions less accurate than those obtained using the Green's function method.

Finally, the solutions for a slab geometry are given for inhomogeneous Robin conditions. The solution is in the form of an infinite series of modified Bessel functions. Worse, the solution for Gaussian irradiance involves an infinite sum of integrals that must be computed numerically. The details of finding the eigenvalues for the problem involve finding the zeros of a transcendental function. As the optical thickness of the slab grows, these eigenvalues decrease very slowly and therefore the number of terms in the series necessary to reach a certain accuracy grows increasingly cumbersome. However, the solutions are complete and can be implemented numerically.

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